

SENSITIVE DEPENDENCE AND TRANSITIVITY OF FUZZIFIED DYNAMICAL SYSTEMS

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ABSTRACT. This paper proves that a set-valued dynamical system is sensitively dependent on initial conditions (resp., \mathcal{F} -sensitive, multi-sensitive) if and only if its g -fuzzification is sensitively dependent on initial conditions (resp., \mathcal{F} -sensitive, multi-sensitive), where \mathcal{F} is a Furstenberg family. As an application, it is shown that there exists a sensitive dynamical system whose g -fuzzification does not have such sensitive dependence for any g in a certain domain. Moreover, a sufficient condition ensuring that the g -fuzzification of every nontrivial dynamical system is not transitive is obtained. These give an answer to a question posed in [16, J. Kupka, Information Sciences, **279** (2014): 642–653].

1. INTRODUCTION

A *dynamical system* is a pair (X, f) , where X is a compact metric space with a metric d and $f : X \rightarrow X$ is a continuous map. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. The complexity of a dynamical system has been a central topic of research since the term of chaos was introduced by Li and Yorke [18] in 1975, known as *Li-Yorke chaos* today. An essential feature of chaos is the impossibility of prediction of its long-term dynamics due to the exponential separation of any two nearby bounded orbits.

Another interesting question about a dynamical system is when orbits from nearby points start to deviate after finite steps. This is also one of the most important features depicting the chaoticity of a system. This concept has been widely studied and is termed as *sensitive dependence on initial conditions* (briefly, *sensitivity*), detailed by Auslander and Yorke [3] and further popularized by Devaney [6]. More precisely, a dynamical system (X, f) is *sensitively dependent* if there exists $\delta > 0$ such that for any $x \in X$ and any $\varepsilon > 0$, there exist $y \in B_d(x, \varepsilon) := \{z \in X : d(x, z) < \varepsilon\}$ and $n \in \mathbb{Z}^+$ satisfying $d(f^n(x), f^n(y)) > \delta$.

In the rest of this Introduction, some notations are used without precise definitions, which will be given in the following section. Given a dynamical system (X, f) , one can obtain two associated systems induced by (X, f) . One is $(\mathcal{K}(X), \bar{f})$ on the hyperspace $\mathcal{K}(X)$ consisting of all nonempty closed subsets of X with the Hausdorff metric. The other is its g -fuzzification system $(\mathbb{F}(X), \tilde{f}_g)$ on the space $\mathbb{F}(X)$ consisting of all upper semicontinuous fuzzy sets with a levelwise metric. The notion of g -fuzzification was introduced by Kupka in [15].

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Afterwards, the connection of dynamical properties among (X, f) , $(\mathcal{K}(X), \bar{f})$ and $(\mathbb{F}(X), \tilde{f}_g)$ has been studied by many researchers.

Bauer and Sigmund [5] studied the interplay of chaos in dynamical systems (individual chaos) with the corresponding set-valued versions (collective chaos). Then, Banks [4] proved that f is weakly mixing if and only if \bar{f} is transitive, which is equivalent to the weakly mixing property of \bar{f} (also see [19]). Guirao et al. [7] proved that \bar{f} has the same type of chaos as f (distributional chaos, Li-Yorke chaos, ω -chaos, topological chaos, specification property, exact Devaney chaos, total Devaney chaos). Hou et al. [9] showed that if f is a non-minimal M -system, then \bar{f} is sensitive. In [8], Gu proved that the sensitivity of \bar{f} implies that f is also sensitive. Lately, Liu et al. [20] gave examples to show that the converse may not hold, i.e., the sensitivity of f does not necessarily imply the sensitivity of \bar{f} , and they proved that if f is a surjective continuous interval map then the sensitivities of \bar{f} and f are equivalent. Li [17] showed that the multi-sensitivity of f and \bar{f} are equivalent properties. Recently, we [24] studied the sensitivity of $(\mathcal{K}(X), \bar{f})$ in Furstenberg families. In particular, we proved that \mathcal{F} -sensitivity of $(\mathcal{K}(X), \bar{f})$ implies that of (X, f) , and the converse is also true if the Furstenberg family \mathcal{F} is a filter [24, Corollary 1, Theorem 4]. For more recent results on the notion of sensitivity, one is referred to [10, 11, 12, 21, 24, 25, 26] and some references therein.

Román-Flores and Chalco-Cano [22] studied some chaotic properties (for example, transitivity, sensitive dependence, periodic density) for Zadeh's extension of a dynamical system. In [14], Kupka investigated the relations between Devaney chaos in the original system and in Zadeh's extension system. Especially, he proved that Zadeh's extension is periodically dense in $\mathbb{F}(X)$ (resp. $\mathbb{F}^\lambda(X)$ for any $\lambda \in (0, 1]$) if and only if \bar{f} is periodically dense in $\mathcal{K}(X)$. Recently, Kupka [15] introduced the notion of g -fuzzification which is a generalization of Zadeh's extension and proved that a dynamical system is continuous if and only if its g -fuzzification system is continuous. In [16], he continued in studying chaotic properties (for example, Li-Yorke chaos, distributional chaos, ω -chaos, transitivity, total transitivity, exactness, sensitive dependence, weakly mixing, mildly mixing, topologically mixing) of g -fuzzification systems and showed that if the g -fuzzification $(\mathbb{F}^1(X), \tilde{f}_g)$ has the property P , then (X, f) also has the property P , where P denotes the following properties: exactness, sensitive dependence, weakly mixing, mildly mixing, or topologically mixing. Meanwhile, he posed the following question:

Question 1. [16] *Does the P -property of (X, f) imply the P -property of $(\mathbb{F}^1(X), \tilde{f}_g)$?*

In this paper, we further investigate the relationships between the sensitivity and the transitivity of set-valued dynamical systems and g -fuzzification through further developing the results in [16]. In this study, we prove that \bar{f} is sensitively dependent if $(\mathbb{F}_0(X), \tilde{f}_g)$ is sensitively dependent. Combining this with [20, Proposition 2.1, Proposition 2.2], we give a negative answer to Question 1 above on sensitivity. Moreover, we obtain the following results:

- (1) $(\mathcal{K}(X), \bar{f})$ is sensitively dependent $\iff (\mathbb{F}^1(X), \tilde{f}_g)$ is sensitively dependent for some $g \in D_m(I)$ with $g^{-1}(1) = \{1\} \iff (\mathbb{F}^1(X), \tilde{f}_g)$ is sensitively dependent for any $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$.
- (2) There exists $g \in D_m(I)$ such that for every nontrivial dynamical system (X, f) , $(\mathbb{F}^1(X), \tilde{f}_g)$ is not transitive (thus, not weakly mixing).

This paper is organized as follows: in Section 2, some basic definitions and notations are introduced. In Section 3, some results obtained in [15, 16] are corrected. Then, in Sections 4 and 5, some preliminary results on the sensitivity, \mathcal{F} -sensitivity, and multi-sensitivity are established and negatively answer Question 1 on sensitivity. Finally, the transitivity is studied in Section 6.

2. BASIC DEFINITIONS AND NOTATIONS

2.1. Furstenberg family, transitivity, and sensitivity. First, recall some basic concepts related to the Furstenberg families (see [2] for more details).

Let \mathcal{P} be the collection of all subsets of \mathbb{Z}^+ . A collection $\mathcal{F} \subset \mathcal{P}$ is called a *Furstenberg family* if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ together imply $F_2 \in \mathcal{F}$. A family \mathcal{F} is *proper* if it is a proper subset of \mathcal{P} , i.e. neither empty nor the whole \mathcal{P} . Throughout this paper, all Furstenberg families are proper. It is clear that a family \mathcal{F} is proper if and only if $\mathbb{Z}^+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Let \mathcal{F}_{inf} be a Furstenberg family of all infinite subsets of \mathbb{Z}^+ . For a family \mathcal{F} , its *dual family* is

$$\kappa\mathcal{F} = \{F \in \mathcal{P} : \mathbb{Z}^+ \setminus F \notin \mathcal{F}\}.$$

It is easy to verify that $\kappa\mathcal{F}$ is a Furstenberg family, and is proper if \mathcal{F} is so. For Furstenberg families \mathcal{F}_1 and \mathcal{F}_2 , let $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. A Furstenberg family \mathcal{F} is a *filter* if \mathcal{F} is proper and $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$.

For $U, V \subset X$, define the *return time set from U to V* as $N(U, V) = \{n \in \mathbb{Z}^+ : f^n(U) \cap V \neq \emptyset\}$. In particular, $N(x, V) = \{n \in \mathbb{Z}^+ : f^n(x) \in V\}$ for $x \in X$.

A dynamical system (X, f) is

- (1) *transitive* if for every pair of nonempty open subsets U, V of X , $N(U, V) \neq \emptyset$;
- (2) *(topologically) weakly mixing* if $(X \times X, f \times f)$ is transitive;
- (3) *mixing* if for every pair of nonempty open subsets U, V of X , there exists $N \in \mathbb{N}$ such that $[N, +\infty) \subset N(U, V)$.

The “largeness” of the time set where sensitivity emerges can be regarded as a measure of how sensitive a system is. For this reason, Moothathu [21] proposed three stronger forms of sensitivity: syndetic sensitivity, cofinite sensitivity (also called strong sensitivity in [1]), and multi-sensitivity. Then, Tan and Zhang [23] introduced a more general description of sensitivity by using Furstenberg families.

Definition 2.1. [17, 21, 23] Let (X, f) be a system and \mathcal{F} be a Furstenberg family.

- (1) (X, f) is *multi-sensitive* if there exists $\varepsilon > 0$ (multi-sensitive constant) such that for any $k \in \mathbb{N}$ and nonempty open subsets $U_1, \dots, U_k \subset X$, $\bigcap_{i=1}^k \{n \in \mathbb{Z}^+ : \text{diam}(f^n(U_i)) > \varepsilon\} \neq \emptyset$, i.e., there exists $n \in \mathbb{Z}^+$ such that $\text{diam}(f^n(U_i)) > \varepsilon$ holds for all $i = 1, \dots, k$, where $\text{diam}(\cdot)$ denotes the diameter of a given set.
- (2) (X, f) is *\mathcal{F} -sensitive* if there exists $\varepsilon > 0$ (\mathcal{F} -sensitive constant) such that for any nonempty open subset $U \subset X$, $\{n \in \mathbb{Z}^+ : \text{diam}(f^n(U)) > \varepsilon\} \in \mathcal{F}$.

2.2. Set-valued dynamical system. Let $\mathcal{K}(X)$ be the hyperspace on X , i.e., the space of nonempty compact subsets of X with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max \left\{ \max_{x \in A} \min_{y \in B} d(x, y), \max_{y \in B} \min_{x \in A} d(x, y) \right\}$$

for any $A, B \in \mathcal{K}(X)$. Clearly, $(\mathcal{K}(X), d_H)$ is a compact metric space. The system (X, f) induces a set-valued dynamical system $(\mathcal{K}(X), \bar{f})$, where $\bar{f} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$

is defined as $\overline{f}(A) = f(A)$ for any $A \in \mathcal{K}(X)$. For any finite collection A_1, \dots, A_n of nonempty subsets of X , let

$$\langle A_1, \dots, A_n \rangle = \left\{ A \in \mathcal{K}(X) : A \subset \bigcup_{i=1}^n A_i, A \cap A_i \neq \emptyset \text{ for all } i = 1, \dots, n \right\}.$$

It follows from [13] that the topology on $\mathcal{K}(X)$ given by the metric d_H is same as the Vietoris or finite topology, which is generated by a basis consisting of all sets of the following form:

$$\langle U_1, \dots, U_n \rangle, \text{ where } U_1, \dots, U_n \text{ are an arbitrary finite collection of nonempty open subsets of } X.$$

2.3. g -fuzzification. A fuzzy set A in space X is a function $A : X \rightarrow I$, where $I = [0, 1]$. Given a fuzzy set A , its α -cuts (or α -level sets) $[A]_\alpha$ and support $\text{supp}(A)$ are defined respectively by

$$[A]_\alpha = \{x \in X : A(x) \geq \alpha\}, \quad \forall \alpha \in I,$$

and

$$\text{supp}(A) = \overline{\{x \in X : A(x) > 0\}}.$$

Let $\mathbb{F}(X)$ denote the set of all upper semicontinuous fuzzy sets defined on X and set

$$\mathbb{F}^\lambda(X) = \{A \in \mathbb{F}(X) : A(x) \geq \lambda \text{ for some } x \in X\}.$$

Define \emptyset_X as the empty fuzzy set ($\emptyset_X \equiv 0$) in X , and $\mathbb{F}_0(X)$ as the set of all nonempty upper semicontinuous fuzzy sets. Since the Hausdorff metric d_H is measured only between two nonempty closed subsets in X , one can consider the following extension of the Hausdorff metric:

$$d_H(\emptyset, \emptyset) = 0, \text{ and } d_H(\emptyset, A) = d_H(A, \emptyset) = \text{diam} X, \quad \forall A \in \mathcal{K}(X).$$

Using this Hausdorff metric, one can define a *levelwise metric* d_∞ on $\mathbb{F}(X)$ by

$$d_\infty(A, B) = \sup \{d_H([A]_\alpha, [B]_\alpha) : \alpha \in (0, 1]\}, \quad \forall A, B \in \mathbb{F}(X).$$

It is well known that the spaces $(\mathbb{F}(X), d_\infty)$ and $(\mathbb{F}^1(X), d_\infty)$ are complete, but not compact and not separable (see [15] and references therein).

A fuzzy set $A \in \mathbb{F}(X)$ is *piecewise constant*, if there exists a finite number of sets $D_i \subset X$ such that $\bigcup \overline{D_i} = X$ and $A|_{\text{int} \overline{D_i}}$ is constant. A can be represented by a sequence of closed subsets $\{A_1, A_2, \dots, A_k\} \subset X$ and an increasing sequence of reals $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subset (0, 1]$ if

$$[A]_\alpha = A_{i+1}, \text{ whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

Kupka [14] proved that the set of all piecewise constant upper continuous maps is dense in $\mathbb{F}(X)$. Then, in [15] he introduced the notion of g -fuzzification to generalize Zadeh's extension.

Zadeh's extension of a dynamical system (X, f) is a map $\tilde{f} : \mathbb{F}(X) \rightarrow \mathbb{F}(X)$ defined by

$$\tilde{f}(A)(x) = \sup \{A(y) : y \in f^{-1}(x)\}, \text{ for any } A \in \mathbb{F}(X) \text{ and any } x \in X.$$

Denote by $D_m(I)$ the set of all nondecreasing right-continuous functions $g : I \rightarrow I$ with $g(0) = 0$ and $g(1) = 1$. For a dynamical system (X, f) and for any $g \in D_m(I)$, define a map $\tilde{f}_g : \mathbb{F}(X) \rightarrow \mathbb{F}(X)$ by

$$\tilde{f}_g(A)(x) = \sup \{g(A(y)) : y \in f^{-1}(x)\}, \text{ for any } A \in \mathbb{F}(X) \text{ and any } x \in X,$$

which is called the g -fuzzification of the dynamical system (X, f) . Clearly, $\tilde{f} = \tilde{f}_{\text{id}_I}$.

Also, define the α -cut $[A]_\alpha^g$ of a fuzzy set $A \in \mathbb{F}(X)$ with respect to $g \in D_m(I)$ by

$$[A]_\alpha^g = \{x \in \text{supp}(A) : g(A(x)) \geq \alpha\}.$$

In [15, 16], Kupka claimed the following:

Lemma 2.1. [15, Lemma 3] *Let (X, f) be a dynamical system, and $g \in D_m(I)$. Then, for any $A \in \mathbb{F}_0(X)$ and any $\alpha \in (0, 1]$, $f([A]_\alpha^g) = [\tilde{f}_g(A)]_\alpha$.*

Lemma 2.2. [15, Lemma 5] *Let $g \in D_m(I)$, $A \in \mathbb{F}_0(X)$, and $\alpha \in (0, 1]$. If $[A]_\alpha^g \neq \emptyset$, then there exists $c \in (0, 1]$ such that $[A]_\alpha^g = [A]_c$.*

Lemma 2.3. [16, Lemma 6] *Let (X, f) be a dynamical system, and $g \in D_m(I)$. Then, for any $A \in \mathbb{F}(X)$ and any $\alpha \in [0, 1]$, $f([A]_\alpha) = [\tilde{f}_g(A)]_{g(\alpha)}$.*

3. SOME REMARKS AND NEW RESULTS

Firstly, we use an example to show that the proof of Lemma 2.2 given in [14] and the above-stated Lemma 2.3 do not hold.

Example 3.1. Let $X = [0, 1]$ and $f : X \rightarrow X$ defined by $f(x) = x$ for all $x \in [0, 1]$. Define $g : [0, 1] \rightarrow [0, 1]$ by

$$g(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}], \\ 1, & x \in [\frac{1}{2}, 1], \end{cases}$$

and take $A \in \mathbb{F}(X)$ with $A = f$. It is easy to see that

$$f([A]_1) = f(\{1\}) = \{1\},$$

and

$$[\tilde{f}_g(A)]_{g(1)} = \{x \in [0, 1] : \tilde{f}_g(A)(x) \geq 1\} = \{x \in [0, 1] : g(A(x)) \geq 1\} = [1/2, 1].$$

So, the above-stated Lemma 2.3 does not hold. In the proof of Lemma 2.2 given in [9], it was claimed that for any $c \in (0, 1]$ with $g(c) > 0$, $[A]_{g(c)}^g = [A]_c$. However, let us simply choose $c = 1$. It can be verified that $[A]_{g(1)}^g = \{x \in [0, 1] : g(A(x)) \geq 1\} = [1/2, 1] \neq \{1\} = [A]_1$.

For any $g \in D_m(I)$, the right-continuity of g implies that $\min g^{-1}([x, 1])$ exists for any $x \in [0, 1]$. Since g is nondecreasing, $\min g^{-1}([x, 1]) > 0$ holds for any $x \in (0, 1]$. Define $\xi_g : [0, 1] \rightarrow [0, 1]$ by $\xi_g(x) = \min g^{-1}([x, 1])$ for any $x \in [0, 1]$. Clearly, ξ_g is nondecreasing.

Next, we give a correct statement and proof Lemma 2.2.

Lemma 3.1. *Let $g \in D_m(I)$, $A \in \mathbb{F}_0(X)$, and $\alpha \in (0, 1]$. Then, there exists $c \in (0, 1]$ such that $[A]_\alpha^g = [A]_{\xi_g(\alpha)}$.*

Proof. Because g is nondecreasing, it follows that

$$[A]_\alpha^g = \{x \in \text{supp}(A) : g(A(x)) \geq \alpha\} = \{x \in \text{supp}(A) : A(x) \geq \xi_g(\alpha)\} = [A]_{\xi_g(\alpha)}.$$

□

Proposition 3.1. *Let (X, f) be a dynamical system, $g \in D_m(I)$, and \tilde{f}_g be the g -fuzzification of f . Then, for any $n \in \mathbb{N}$, any $A \in \mathbb{F}(X)$, and any $\alpha \in (0, 1]$, $\left[(\tilde{f}_g)^n(A)\right]_\alpha = f^n([A]_{\xi_g^n(\alpha)})$. In particular, for any $B \in \mathcal{K}(X)$, $\left[(\tilde{f}_g)^n(\chi_B)\right]_\alpha = f^n(B)$.*

Proof. Applying Lemma 2.1 and Lemma 3.1, it follows that $\left[(\tilde{f}_g)(A)\right]_\alpha = f([A]_\alpha^g) = f^n([A]_{\xi_g^n(\alpha)})$. Applying mathematical induction, it is not difficult to verify that the proposition is true. \square

Proposition 3.2. *Let (X, f) be a dynamical system, and $g \in D_m(I)$. Then, for any $A \in \mathbb{F}(X)$ and any $\alpha \in [0, 1]$, $f([A]_\alpha) \subset \left[\tilde{f}_g(A)\right]_{g(\alpha)}$.*

Proof. For any $x \in f([A]_\alpha)$, there exists $y \in [A]_\alpha$ such that $x = f(y)$. It is easy to see that $\tilde{f}_g(A)(x) = g(A(y)) \geq g(\alpha)$. So, $x \in \left[\tilde{f}_g(A)\right]_{g(\alpha)}$. \square

Lemma 3.2. *Let (X, f) be a dynamical system, and $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$. Then, for any $n \in \mathbb{N}$ and any $A \in \mathbb{F}(X)$, $\left[(\tilde{f}_g)^n(A)\right]_1 = f^n([A]_1)$.*

Proof. Applying Proposition 3.1, noting that $g^{-1}(1) = \{1\}$, it is easy to verify that the lemma is true. \square

4. SENSITIVITY OF g -FUZZIFICATION

This section is devoted to studying the sensitivity of the g -fuzzification systems.

Proposition 4.1. *Let (X, f) be a dynamical system, $A \in \mathcal{K}(X)$ and $\{B_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{K}(X)$. If there exists $\xi > 0$ such that for any $\lambda \in \Lambda$, $d_H(A, B_\lambda) < \xi$, then $d_H(A, \overline{\bigcup_{\lambda \in \Lambda} B_\lambda}) \leq \xi$.*

Proof. It is easy to see that for any $\lambda_0 \in \Lambda$,

$$\begin{aligned} \sup_{x \in A} \inf_{y \in \bigcup_{\lambda \in \Lambda} B_\lambda} d(x, y) &\leq \sup_{x \in A} \inf_{y \in \bigcup_{\lambda \in \Lambda} B_\lambda} d(x, y) \\ (1) \quad &\leq \sup_{x \in A} \inf_{y \in B_{\lambda_0}} d(x, y) \leq d_H(A, B_{\lambda_0}) < \xi. \end{aligned}$$

Now, for any $y \in \overline{\bigcup_{\lambda \in \Lambda} B_\lambda}$, consider the following two cases:

- (a) if $y \in \bigcup_{\lambda \in \Lambda} B_\lambda$, then there exists $\lambda \in \Lambda$ such that $y \in B_\lambda$. Thus, $\inf_{x \in A} d(x, y) \leq d_H(A, B_\lambda) < \xi$;
- (b) if $y \in \overline{\bigcup_{\lambda \in \Lambda} B_\lambda} \setminus \bigcup_{\lambda \in \Lambda} B_\lambda$, then for any $n \in \mathbb{N}$, there exist $\lambda_n \in \Lambda$ and $z \in B_{\lambda_n}$ such that $d(y, z) < 1/n$. According to the definition of $d_H(A, B_{\lambda_n})$, there exists $x_n \in A$ such that $d(x_n, z) \leq d_H(A, B_{\lambda_n}) < \xi$. So,

$$\inf_{x \in A} d(x, y) \leq d(x_n, y) \leq d(x_n, z) + d(z, y) < \xi + \frac{1}{n}.$$

This implies that $\sup_{y \in \overline{\bigcup_{\lambda \in \Lambda} B_\lambda}} \inf_{x \in A} d(x, y) \leq \xi$. Combining this with (1), it follows that $d_H(A, \overline{\bigcup_{\lambda \in \Lambda} B_\lambda}) \leq \xi$. \square

Theorem 4.1. *Let (X, f) be a dynamical system and $g \in D_m(I)$. If $(\mathbb{F}_0(X), \tilde{f}_g)$ is sensitively dependent, then $(\mathcal{K}(X), \bar{f})$ is sensitively dependent.*

Proof. Let $\varepsilon > 0$ be a sensitive constant of \tilde{f}_g . Given any fixed $A \in \mathcal{K}(X)$ and any $\delta > 0$, noting that $\chi_A \in \mathbb{F}_0(X)$, the sensitivity of \tilde{f}_g implies that there exist $B \in \mathbb{F}_0(X)$ with $d_\infty(\chi_A, B) < \frac{\delta}{2}$ and $n \in \mathbb{Z}^+$ such that

$$(2) \quad d_\infty((\tilde{f}_g)^n(\chi_A), (\tilde{f}_g)^n(B)) > \varepsilon.$$

Applying Proposition 3.1, it follows that for any $\alpha \in (0, 1]$, there exists $\xi_g^n(\alpha) \in (0, 1]$ such that

$$\left[(\tilde{f}_g)^n(\chi_A) \right]_\alpha = \bar{f}^n(A), \text{ and } \left[(\tilde{f}_g)^n(B) \right]_\alpha = \bar{f}^n([B]_{\xi_g^n(\alpha)}).$$

This, together with (2), implies that there exists $\alpha_0 \in (0, 1]$ such that

$$d_H \left(\left[(\tilde{f}_g)^n(\chi_A) \right]_{\alpha_0}, \left[(\tilde{f}_g)^n(B) \right]_{\alpha_0} \right) = d_H \left(\bar{f}^n(A), \bar{f}^n([B]_{\xi_g^n(\alpha_0)}) \right) > \varepsilon.$$

Clearly,

$$d_H(A, [B]_{\xi_g^n(\alpha_0)}) = d_H([\chi_A]_{\xi_g^n(\alpha_0)}, [B]_{\xi_g^n(\alpha_0)}) \leq d_\infty(\chi_A, B) < \delta.$$

Since A and δ are arbitrary, it is concluded that \bar{f} is sensitively dependent. \square

Theorem 4.2. *Let (X, f) be a dynamical system. Then, the following statements are equivalent:*

- (1) $(\mathcal{K}(X), \bar{f})$ is sensitively dependent;
- (2) $(\mathbb{F}^1(X), \tilde{f}_g)$ is sensitively dependent for some $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$;
- (3) $(\mathbb{F}^1(X), \tilde{f}_g)$ is sensitively dependent for any $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$.

Proof. (2) \implies (1). This follows immediately from Theorem 4.1.

(1) \implies (3). Let $\varepsilon > 0$ be a sensitive constant of \bar{f} and fix any $g \in D_m(I)$. For any $A \in \mathbb{F}^1(X)$ and any $\delta > 0$, clearly, $[A]_1 \in \mathcal{K}(X)$. Since \bar{f} is sensitive, there exist $C \in \mathcal{K}(X)$ with $d_H([A]_1, C) < \frac{\delta}{4}$ and $n \in \mathbb{Z}^+$ such that $d_H(\bar{f}^n([A]_1), \bar{f}^n(C)) > \varepsilon$. The continuity of \bar{f} implies that there exists $0 < \xi < \delta/4$ such that for any $F \in \{F_1 \in \mathcal{K}(X) : d_H(F_1, C) \leq \xi\}$, $d_H(\bar{f}^n(C), \bar{f}^n(F)) < \frac{d_H(\bar{f}^n([A]_1), \bar{f}^n(C)) - \varepsilon}{2}$. Set $Q = \{y \in X : \inf_{x \in C} d(x, y) < \xi\} \in \mathcal{K}(X)$. Clearly, $d_H(Q, C) \leq \xi$. So,

$$(3) \quad d_H(\bar{f}^n([A]_1), \bar{f}^n(Q)) \geq d_H(\bar{f}^n([A]_1), \bar{f}^n(C)) - d_H(\bar{f}^n(C), \bar{f}^n(Q)) > \varepsilon.$$

Take $X_1 = X$, $\alpha_1 = \max_{x \in X_1} A(x) = 1$ and $D_1 = A^{-1}(\alpha_1) \cap X_1 = [A]_1$. Define inductively $X_{i+1} = X_i \setminus \{y \in X : \inf_{x \in D_i} d(x, y) < \frac{\delta}{4}\}$, $\alpha_{i+1} = \max_{x \in X_{i+1}} A(x)$ and $D_{i+1} = A^{-1}(\alpha_{i+1}) \cap X_{i+1}$ for $i \geq 1$. The compactness of X implies that there exists $k \in \mathbb{N}$ such that $X_{k+1} = \emptyset$. Denote $U_i = \overline{\bigcup_{j=1}^i \{y \in X : \inf_{x \in D_j} d(x, y) < \frac{\delta}{4}\}}$ for $i = 1, \dots, k$ and take a piecewise constant fuzzy set \mathcal{A} satisfying $[\mathcal{A}]_\alpha = U_i$ for $\alpha \in (\alpha_{i+1}, \alpha_i]$. It follows from the proof of [14, Lemma 1] that $d_\infty(A, \mathcal{A}) < \frac{\delta}{4}$.

Next, take another piecewise constant fuzzy set $E \in \mathbb{F}^1(X)$ such that

$$[E]_\alpha = \begin{cases} Q, & \alpha \in (\frac{\alpha_2 + \alpha_1}{2}, \alpha_1], \\ U_1 \cup Q, & \alpha \in (\alpha_2, \frac{\alpha_2 + \alpha_1}{2}], \\ U_i \cup Q, & \alpha \in (\alpha_{i+1}, \alpha_i], i \in \{2, \dots, k\}. \end{cases}$$

It can be verified that

$$d_H(Q, U_1) \leq d_H(Q, C) + d_H(C, [A]_1) + d_H([A]_1, U_1) \leq \xi + \frac{\delta}{4} + \frac{\delta}{4} < \frac{3\delta}{4}.$$

From this, it follows that $d_\infty(\mathcal{A}, E) < \frac{3\delta}{4}$, so that

$$d_\infty(A, E) \leq d_\infty(A, \mathcal{A}) + d_\infty(\mathcal{A}, E) < \delta.$$

Now, applying Lemma 3.2 and (3), one has

$$\begin{aligned} d_\infty((\tilde{f}_g)^n(A), (\tilde{f}_g)^n(E)) &\geq d_H\left(\left[(\tilde{f}_g)^n(A)\right]_1, \left[(\tilde{f}_g)^n(E)\right]_1\right) \\ &= d_H\left(\bar{f}^n([A]_1), \bar{f}^n([E]_1)\right) = d_H\left(\bar{f}^n([A]_1), \bar{f}^n(Q)\right) > \varepsilon. \end{aligned}$$

(3) \implies (2). It is obvious. \square

Theorem 4.2, together with the fact that $\tilde{f} = \tilde{f}_{\text{id}_I}$, yields the following result.

Corollary 4.1. *Let (X, f) be a dynamical system. Then $(\mathcal{K}(X), \bar{f})$ is sensitively dependent if and only if $(\mathbb{F}^1(X), \tilde{f})$ is sensitively dependent.*

To close this section, we apply Theorem 4.1 and [20, Proposition 2.1, Proposition 2.2] to construct a counterexample which gives a negative answer to Question 1 above.

Example 4.1. Let \mathbb{R}/\mathbb{Z} be the domain of the unite circle \mathbb{S}^1 . Define a metric d by $d(a, b) = \min\{|a - b|, 1 - |a - b|\}$. Then, the rigid rotation $R_\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by a real number α is given by

$$R_\alpha(t) = t + \alpha \pmod{1}, \text{ for all } t \in \mathbb{R}.$$

Corresponding to the irrational α , the Denjoy homeomorphism $D_\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an orientation preserving homeomorphism of the circle characterized by the following properties:

- (1) the rotational number of D_α is α ;
- (2) there is a Cantor set $C_\alpha \subset \mathbb{S}^1$ such that $D_\alpha|_{C_\alpha}$ is minimal.

In [20, Proposition 2.1, Proposition 2.2], Liu et al. proved that $(C_\alpha, D_\alpha|_{C_\alpha})$ is sensitively dependent but its set-valued dynamical system $(\mathcal{K}(C_\alpha), \overline{D_\alpha|_{C_\alpha}})$ is not sensitively dependent. This, together with Theorem 4.1, implies that for every $g \in D_m(I)$, the g -fuzzification of $(C_\alpha, D_\alpha|_{C_\alpha})$ is not sensitively dependent. This shows that the answer to Question 1 is negative.

5. \mathcal{F} -SENSITIVITY AND MULTI-SENSITIVITY OF g -FUZZIFICATION

As an extension of the last section, this section is devoted to studying \mathcal{F} -sensitivity and multi-sensitivity of g -fuzzification.

Theorem 5.1. *Let (X, f) be a system and \mathcal{F} be a Furstenberg family. Then, the following statements are equivalent:*

- (1) $(\mathcal{K}(X), \bar{f})$ is \mathcal{F} -sensitive;
- (2) $(\mathbb{F}^1(X), \tilde{f}_g)$ is \mathcal{F} -sensitive for some $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$;
- (3) $(\mathbb{F}^1(X), \tilde{f}_g)$ is \mathcal{F} -sensitive for any $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$.

Proof. Similarly to the proof of Theorem 4.1, it can be verified that (3) \implies (2) \implies (1). It suffices to check that (1) \implies (3).

Let $\varepsilon > 0$ be a \mathcal{F} -sensitive constant of \bar{f} and fix any $g \in D_m(I)$. For any $A \in \mathbb{F}^1(X)$ and any $\delta > 0$, the \mathcal{F} -sensitivity of \bar{f} implies that there exists $F \in \mathcal{F}$ such that for any $n \in F$, there exists $C \in \mathcal{K}(X)$ with $d_H([A]_1, C) < \frac{\delta}{4}$ satisfying $d_H(\bar{f}^n([A]_1), \bar{f}^n(C)) > \frac{\varepsilon}{2}$. Similarly to the proof of Theorem 4.2, it follows that

there exists $E \in \mathbb{F}^1(X)$ such that $d_\infty(A, E) < \delta$ and $d_\infty((\tilde{f}_g)^n(A), (\tilde{f}_g)^n(E)) > \varepsilon/2$. This implies that $F \subset \left\{ n \in \mathbb{Z}^+ : \text{diam}((\tilde{f}_g)^n(B_{d_\infty}(A, \delta))) > \varepsilon/2 \right\} \in \mathcal{F}$. So, $(\mathbb{F}^1(X), \tilde{f}_g)$ is \mathcal{F} -sensitive, as A and δ are arbitrary. \square

Combining Theorem 5.1 with [24, Theorem 4], one can immediately obtain the following.

Corollary 5.1. *Let (X, f) be a system and \mathcal{F} be a filter. Then, the following statements are equivalent:*

- (1) (X, f) is \mathcal{F} -sensitive;
- (2) $(\mathcal{K}(X), \bar{f})$ is \mathcal{F} -sensitive;
- (3) $(\mathbb{F}^1(X), \tilde{f}_g)$ is \mathcal{F} -sensitive for some $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$;
- (4) $(\mathbb{F}^1(X), \tilde{f}_g)$ is \mathcal{F} -sensitive for any $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$.

Slightly modifying the proofs of Theorem 4.2 and Theorem 5.1 and applying [17, Theorem 3.2, Theorem 3.3], one can prove the following.

Theorem 5.2. *Let (X, f) be a system. Then, the following statements are equivalent:*

- (1) (X, f) is multi-sensitive;
- (2) $(\mathcal{K}(X), \bar{f})$ is multi-sensitive;
- (3) $(\mathbb{F}^1(X), \tilde{f}_g)$ is multi-sensitive for some $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$;
- (4) $(\mathbb{F}^1(X), \tilde{f}_g)$ is multi-sensitive for any $g \in D_m(I)$ with $g^{-1}(1) = \{1\}$.

6. TRANSITIVITY OF g -FUZZIFICATION

For the weakly mixing property of g -fuzzification, we have the following result:

Theorem 6.1. *Let (X, f) be a dynamical system, and $g \in D_m(I)$. If $(\mathbb{F}^1(X), \tilde{f}_g)$ is transitive, then $(\mathcal{K}(X), \bar{f})$ is weakly mixing.*

Proof. Applying [4, Theorem 2], it suffices to prove that \bar{f} is transitive.

For any pair of nonempty open subsets $U, V \subset \mathcal{K}(X)$, there exist $A \in U$, $B \in V$ and $\delta > 0$ such that $B_{d_H}(A, \delta) \subset U$ and $B_{d_H}(B, \delta) \subset V$. Noting that $B_{d_\infty}(\chi_A, \delta)$ and $B_{d_\infty}(\chi_B, \delta)$ are nonempty subsets of $\mathbb{F}^1(X)$, since \tilde{f}_g is transitive, there exists $n \in \mathbb{Z}^+$ such that $(\tilde{f}_g)^n(B_{d_\infty}(\chi_A, \delta)) \cap B_{d_\infty}(\chi_B, \delta) \neq \emptyset$. Then, there exists a point $F_1 \in B_{d_\infty}(\chi_A, \delta)$ such that $(\tilde{f}_g)^n(F_1) \in B_{d_\infty}(\chi_B, \delta)$. This implies that, for any $\alpha \in (0, 1]$,

$$(4) \quad d_H\left(\left[(\tilde{f}_g)^n(F_1)\right]_\alpha, B\right) < \delta.$$

In particular, applying Proposition 3.1, it follows that there exists $\xi \in (0, 1]$ such that

$$(5) \quad \left[(\tilde{f}_g)^n(F_1)\right]_{1/2} = \bar{f}^n([F_1]_\xi).$$

Since $F_1 \in B_{d_\infty}(\chi_A, \delta)$, it is easy to see that $d_H(A, [F_1]_\xi) < \delta$, i.e., $[F_1]_\xi \in B_{d_H}(A, \delta) \subset U$. Combining this with (4) and (5), it follows that

$$\bar{f}^n([F_1]_\xi) \in \bar{f}^n(U) \cap V \neq \emptyset.$$

\square

Being the end of this section, we shall prove that there exists $g \in D_m(I)$ such that the g -fuzzification system of every nontrivial dynamical system is not transitive, giving a partial answer to Question 1. The following lemma is obvious.

Lemma 6.1. *A dynamical system (X, f) is transitive if and only if for every pair of nonempty open subsets U, V of X , $N(U, V) \in \mathcal{F}_{inf}$.*

Theorem 6.2. *Let (X, d) be a nontrivial metric space and $g \in D_m(I)$ satisfying that there exist $z \in (0, 1]$ and $m \in \mathbb{N}$ such that $\xi_g(z) \neq z$ and $\xi_g^{m+1}(z) = \xi_g^m(z)$. Then, for any $f \in \mathcal{C}(X)$, its g -fuzzification system $(\mathbb{F}^1(X), \tilde{f}_g)$ is not transitive, where $\mathcal{C}(X)$ is the set of all continuous self-maps defined on X .*

Proof. Fix two distinct points $a, b \in X$, as X is nontrivial. To prove this theorem, consider two cases as follows:

Case 1. $\xi_g(z) > z$. Since ξ_g is nondecreasing, applying mathematical induction, it follows that for any $j \in \mathbb{N}$,

$$(6) \quad \xi_g^{j+1}(z) \geq \xi_g^j(z) \geq \xi_g(z) > z.$$

Set

$$E_1 = \left\{ x \in X : d(x, a) \leq \frac{d(a, b)}{8} \right\},$$

and

$$E_2 = \left\{ x \in X : d(x, b) \leq \frac{d(a, b)}{8} \right\}.$$

Take two fuzzy sets $E, G \in \mathbb{F}^1(X)$ such that

$$E(x) = \begin{cases} 1, & x \in E_1, \\ z, & x \in E_2, \\ 0, & x \in X \setminus (E_1 \cup E_2), \end{cases}$$

and

$$G(x) = \begin{cases} 1, & x \in E_2, \\ z, & x \in E_1, \\ 0, & x \in X \setminus (E_1 \cup E_2). \end{cases}$$

Let

$$\eta = \inf \{d(x, y) : x \in E_1, y \in E_2\} \geq \frac{3}{4}d(a, b)$$

and

$$\mathcal{U} = \left\{ F \in \mathbb{F}^1(X) : d_\infty(F, E) < \frac{\eta}{4} \right\},$$

$$\mathcal{V} = \left\{ F \in \mathbb{F}^1(X) : d_\infty(F, G) < \frac{\eta}{4} \right\}.$$

Since $d_\infty(E, G) \geq d_H([E]_1, [G]_1) = d_H(E_1, E_2) \geq \eta$, then $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Now, we claim that for any $n > m$, $(\tilde{f}_g)^n(\mathcal{U}) \cap \mathcal{V} = \emptyset$.

In fact, if there exist some $n > m$ such that $(\tilde{f}_g)^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$, then there exists $P \in \mathcal{U}$ such that $(\tilde{f}_g)^n(P) \in \mathcal{V}$. This implies that for any $\alpha \in (0, 1]$,

$$d_H([\tilde{f}_g^n(P)]_\alpha, [G]_\alpha) < \frac{\eta}{4}.$$

In particular, applying (6) and Proposition 3.1, it follows that

$$d_H([\tilde{f}_g^n(P)]_z, [G]_z) = d_H(\bar{f}^n([P]_{\xi_g^n(z)}), E_1 \cup E_2) < \frac{\eta}{4},$$

and

$$\begin{aligned} d_H([\tilde{f}_g]^n(P)]_{\xi_g(z)}, [G]_{\xi_g(z)}) &= d_H(\bar{f}^n([P]_{\xi_g^{n+1}(z)}), E_2) \\ &= d_H(\bar{f}^n([P]_{\xi_g^n(z)}), E_2) < \frac{\eta}{4}. \end{aligned}$$

So,

$$\begin{aligned} \frac{\eta}{2} &> d_H(\bar{f}^n([P]_{\xi_g^n(z)}), E_1 \cup E_2) + d_H(\bar{f}^n([P]_{\xi_g^n(z)}), E_2) \\ &\geq d_H(E_1 \cup E_2, E_2) = \max_{x \in E_1 \cup E_2} \inf_{y \in E_2} d(x, y) \\ &= \max_{x \in E_1} \inf_{y \in E_2} d(x, y) \geq \eta, \end{aligned}$$

which is a contradiction as $\eta > 0$.

Case 2. $\xi_g(z) < z$. Similarly to the proof of Case 1, it can be verified that there exist nonempty open subsets \mathcal{U}, \mathcal{V} of $\mathbb{F}^1(X)$ such that for any $n > m$, $(\tilde{f}_g)^n(\mathcal{U}) \cap \mathcal{V} = \emptyset$.

Summing up Case 1 and Case 2, applying Lemma 6.1, it follows that $(\mathbb{F}^1(X), \tilde{f}_g)$ is not transitive. \square

Remark 1. Choose $g : I \rightarrow I$ as

$$g(x) = \begin{cases} 0, & x = 0, \\ 1 - \frac{1}{2^n}, & x \in [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}), n \in \mathbb{N}, \\ 1, & x = 1. \end{cases}$$

It can be verified that $g \in D_m(I)$, and

$$\xi_g(x) = \begin{cases} 0, & x = 0, \\ 1 - \frac{1}{2^{n+1}}, & x \in (1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}], n \in \mathbb{N}, \\ 1, & x = 1. \end{cases}$$

Clearly, $z = 1/4$ satisfies that $\xi_g(z) = 1/2 \neq z$, and $\xi_g^n(z) = 1/2$ for all $n \geq 2$. This, together with Theorem 6.2, implies that the answer to Question 1 is negative.

7. CONCLUSIONS

In this paper, we present a systematic study of the sensitivity of g -fuzzification. Firstly, we prove that $(\mathcal{K}(X), \bar{f})$ is sensitively dependent if $(\mathbb{F}_0(X), \tilde{f}_g)$ is so (see Theorem 4.1). This, together with Example 4.1, gives a negative answer to Question 1 posed in [16]. Then, we reveal some characteristics ensuring that $(\mathcal{K}(X), \bar{f})$ is sensitive, \mathcal{F} -sensitive, or multi-sensitive (see Theorem 4.2, Theorem 5.1, and Theorem 5.2, respectively). Moreover, we show that $(\mathcal{K}(X), \bar{f})$ is weakly mixing provided that $(\mathbb{F}^1(X), \tilde{f}_g)$ is transitive. Finally, we prove that there exists $g \in D_m(I)$ such that for any dynamical system (X, f) , $(\mathbb{F}^1(X), \tilde{f}_g)$ is not transitive (thus, not weakly mixing).

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